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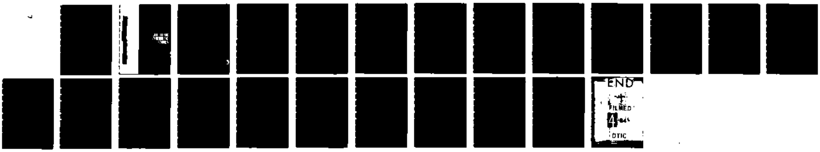

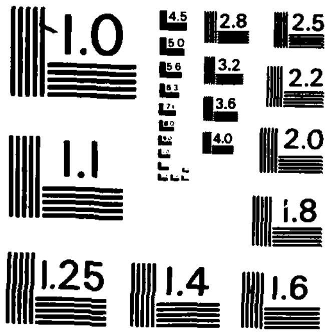

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Research Report CCS 449

THE EXTREME POINT CHARACTERIZATIONS OF SEMI-INFINITE DUAL NON-ARCHIMEDEAN BALLS

by

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ABSTRACT

The extreme point characterization of the (ℓ')-ball of a generalized finite sequence space by Kortanek and Strojwas was accomplished only for real scalars and by continuity considerations. [illegible] ~~We~~ shows that no topology or continuity is needed as in Kortanek-Strojwas and that the characterization extends to weighted (ℓ')-balls with any ordered scalar field. We show a Chebyshev ball theorem is shown to be false since ~~they have~~ has no extreme points. Via generalizing the LIEP (Linear Independence with Extreme Points) theorem, useful projections of the ball are proved convex hulls of their extreme points.

KEY WORDS

Semi-infinite programming

Non-Archimedean programming

Weighted (ℓ')-balls

Chebyshev balls

THE EXTREME POINT CHARACTERIZATIONS OF SEMI-INFINITE

DUAL NON-ARCHIMEDEAN BALLS

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1. Introduction

In 1951 Charnes introduced non-Archimedean field extensions into linear programming as part of his non-Archimedean Simplex method [1] which solved the degeneracy problem and thereby provided the first rigorous algorithm for solution of linear programming problems. Together with his LIEP Theorem and Opposite Sign Theorem it could be used to extend the major theorems of LP to vector spaces with scalars from any ordered field (e.g. [2] and [3]) without thereby requiring topological considerations as used in separation theorems for convex sets. Although the LIEP Theorem and Opposite Sign Theorem were extended to semi-infinite programming duals in [4], Kortanek and Strojwas in [5] succeeded only in the important case of the real field and by means of continuity considerations to characterize in a similar fashion the extreme points of dual constraints sets additionally constrained to lie in a (non-linear) "(ℓ')-ball" of the generalized finite sequence space.

In this paper we show that no topological or continuity considerations are needed and that the Kortanek-Strojwas characterization holds for the extension to weighted (ℓ')-balls with vector entries from <u>any</u> ordered field. We also prove that the similar theorem for "Chebyshev-balls" is <u>false</u>. In fact the Chebyshev-balls have no extreme points. Via a generalization of the LIEP Theorem of semi-infinite programming, we obtain as corollaries characterization of useful subsets of the Chebyshev ball as convex hulls of their extreme points.

First we define the following sets

(1) $$\Lambda \triangleq \{ \lambda \in F^{(I)} : \sum_i P_i\lambda_i = Q , \quad \lambda \geq 0 \}$$

(2) $$\tilde{\Lambda} \triangleq \{ \lambda \in F^{(I)} : \sum_i P_i\lambda_i = Q , \quad \lambda \geq 0 , \sum_i \lambda_i \leq U \}$$

(3) $$\hat{\Lambda} \triangleq \{ \lambda \in F^{(I)} : \sum_i P_i\lambda_i = Q , \qquad \sum_i |\lambda_i| \leq U \}$$

(4) $$\Lambda^c \triangleq \{ \lambda \in F^{(I)} : \sum_i P_i\lambda_i = Q , \qquad |\lambda_i| \leq U , \ i \in I \}$$

where F is any ordered field; P_i's, Q are m-vector from F^m; I is an index set; $\sum_i$ means the summation is over all non-zero components of λ. $F^{(I)}$ is the generalized finite sequence space of vectors on F with $|I|$ entries, alternately it is the space of functions from I to F with finitely many non-zero entries.

In the following section, we will show that sets $\tilde{\Lambda}$, $\hat{\Lambda}$ are all the convex hull of their extreme points and we will also discuss some properties of their extreme points. The fundamental theorems of this paper are the LIEP Theorem and OS Theorem:

<u>Theorem 1.1</u> (Linear Independence with Extreme Points)

Assume Λ of (1) is non-empty. Then $\lambda \neq 0$ is an extreme point of Λ if and only if $\{ P_i \mid i \in I \}$ is linearly independent.

<u>Theorem</u> 1.2 (Opposite Sign Theorem)

Assume Λ is non-empty. Then the set of extreme points of Λ is non-empty and Λ is the convex hull of its extreme points if and only if $\{ P_i \mid i \in I \}$ has the Opposite Sign Property, (OSP) namely, $\lambda \in F^{(I)}$, $\lambda \neq 0$ and $\sum_i P_i\lambda_i = 0$ imply that some λ_r and λ_s are of opposite sign.

2. The set $\tilde{\Lambda}$

Consider

(5) $$\tilde{\Lambda}'' \triangleq \{(\lambda^*,\lambda) \in F' \times F^{(I)} : \lambda^*\binom{0}{1} + \sum_i \lambda_i \binom{P_i}{1} = \binom{Q}{U}, \ \lambda \geqslant 0, \ \lambda^* \geqslant 0\}$$

For any $\lambda \in \tilde{\Lambda}$, let

$$\lambda^* = U - \sum_i \lambda_i$$

This defines the following mapping:

$$\varphi : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}''$$

where $\varphi(\lambda) = (\lambda^*,\lambda) = (U - \sum_i \lambda_i \ , \lambda)$

Evidently this mapping is 1-1 and of the first degree in λ.

Take $\lambda^1, \lambda^2 \in \tilde{\Lambda}$ and $0 < \theta < 1$. Since

$$U - (\sum_i (\theta\lambda^1 + (1-\theta)\lambda_i^2)) = \theta(U - \sum_i \lambda_i^1) + (1-\theta)(U - \sum_i \lambda_i^2) \ ,$$

we have

$$\varphi(\theta\lambda^1 + (1-\theta)\lambda^2) = \theta\varphi(\lambda^1) + (1-\theta)\varphi(\lambda^2)$$

Conversely,

$$\varphi^{-1}(\theta\varphi(\lambda^1) + \overline{1-\theta}\ \varphi(\lambda^2)) = \varphi^{-1}[\varphi(\theta\lambda^1 + \overline{1-\theta}\lambda^2)] = \theta\lambda^1 + (1-\theta)\lambda^2$$

The following Lemmas are true.

<u>Lemma 2.1</u>

λ is an extreme point of $\tilde{\Lambda}$ if and only if $\varphi(\lambda)$ is an extreme point of $\tilde{\Lambda}''$.

<u>Lemma 2.2</u>

$\tilde{\Lambda}$ is the convex hull of its extreme points if and only if $\tilde{\Lambda}''$ is the convex hull of its extreme points.

<u>Theorem 2.1</u>

If $\tilde{\Lambda}$ is non-empty, then $\tilde{\Lambda}$ is the convex hull of its extreme points.

<u>Proof</u>: Since $\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}P_i\\1\end{pmatrix} : i \in I\right\}$ has the opposite sign property and $\tilde{\Lambda}'' \neq \phi$ because $\tilde{\Lambda} \neq \phi$, by Theorem 1.2 $\tilde{\Lambda}''$ is the convex hull of its extreme points.

In accordance with Lemma 2.2, then $\tilde{\Lambda}$ is the convex hull of its extreme points.

Q.E.D.

<u>Theorem 2.2</u>

Suppose λ is an extreme point of $\tilde{\Lambda}$ and $\lambda \neq 0$.

(i) If $\sum_i \lambda_i = U$, then $\{P_i : \lambda_i > 0\}$ is affinely independent.

(ii) If $\sum_i \lambda_i < U$, then $\{P_i : \lambda_i > 0\}$ is linearly independent.

<u>Proof</u>: Suppose λ is an extreme point of $\tilde{\Lambda}$. By Lemma 2.1 $\varphi(\lambda)$ is an extreme point of $\tilde{\Lambda}''$.

(i) If $\sum_i \lambda_i = U$, then $\lambda^* = U - \sum_i \lambda_i = 0$.

By Theorem 1.1,

$$\left\{\begin{pmatrix}P_i\\1\end{pmatrix} : \lambda_i > 0\right\} \text{ is linearly independent,}$$

i.e.,

$$\{P_i : \lambda_i > 0\} \quad \text{is affinely independent.}$$

(ii) If $\sum_i \lambda_i < U$, then $\lambda^* = U - \sum_i \lambda_i > 0$.

By Theorem 1.1,

$$\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}P_i\\1\end{pmatrix} : \lambda_i > 0\right\} \text{ is linearly independent.}$$

Hence the set $\left\{\begin{pmatrix}P_i\\0\end{pmatrix} : \lambda_i > 0\right\}$ is linearly independent and $\{P_i : \lambda_i > 0\}$ is linearly independent.

Q.E.D.

3. The set $\hat{\Lambda}$

Below we will first discuss the more general set $\hat{\Lambda}_w$ as follows:

$$(6) \qquad \hat{\Lambda}_w \triangleq \{\lambda \in F^{(I)} : \sum_i p_i \lambda_i = Q \ , \ \sum_i w_i |\lambda_i| \leqslant U\}$$

where $w_i > 0$, for all $i \in I$, is called the "weight" of component i and

$|\lambda_i| \triangleq \max(\lambda_i \ , -\lambda_i)$.

Lemma 3.1

The function $g(\rho) \triangleq \sum_i w_i |\lambda_i + \rho\alpha_i|$,

where (i) $w_i > 0$, $\forall i \in I$, (ii) $\{i \in I : \alpha_i \neq 0, \text{ or } \lambda_i \neq 0\}$ is finite and $\alpha_i \neq 0$ for some i, is a non-negative piecewise linear function of $\rho \geqslant 0$, which takes on all values in F between any two values of $g(\rho)$ and takes on arbitrarily large positive values in F.

Proof:

Let

$$I_0 \triangleq \{i : \lambda_i = 0 \ , \ \alpha_i \neq 0\}$$

$$I_{00} \triangleq \{i : \lambda_i \neq 0 \ , \ \alpha_i = 0\}$$

$$I_s \triangleq \{i : \lambda_i \alpha_i > 0\}$$

$$I_d \triangleq \{i : \lambda_i \alpha_i < 0\}$$

Thus $I_0 \cup I_{00} \cup I_s \cup I_d$ is a partition of $\{i \in I : \alpha_i \neq 0 \text{ or } \lambda_i \neq 0\}$.

Then

$$g(\rho) = \sum_i w_i |\lambda_i + \rho\alpha_i|$$

$$= \sum_{i \in I_{00}} w_i |\lambda_i| + \rho \sum_{i \in I_0} w_i |\alpha_i| + \sum_{i \in I_s} (|\lambda_i| + \rho|\alpha_i|) \ +$$

$$+ \sum_{\substack{i \in I_d \\ \rho \leqslant |\lambda_i| / |\alpha_i|}} w_i(|\lambda_i| - \rho|\alpha_i|) + \sum_{\substack{i \in I_d \\ \rho > |\lambda_i| / |\alpha_i|}} w_i(\rho|\alpha_i| - |\lambda_i|)$$

Let $\quad \dfrac{|\lambda_{i_1}|}{|\alpha_{i_1}|} \leqslant \dfrac{|\lambda_{i_2}|}{|\alpha_{i_2}|} \leqslant \ldots \leqslant \dfrac{|\lambda_{i_n}|}{|\alpha_{i_n}|}$

where $\{i_1, \ldots, i_n\} = I_d$. We designate these ratios as $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho_n$.

Thereby we obtain the following expressions for $g(\rho)$.

For

$\rho_o \triangleq 0 \leqslant \rho < \rho_1$:

$$g(\rho) = \sum_{I_{oo} \cup I_s \cup I_d} w_i|\lambda_i| + \rho\{\sum_{I_o \cup I_s} w_i|\alpha_i| - \sum_{I_d} w_i|\alpha_i|\}$$

For

$\rho_k \leqslant \rho < \rho_{k+1}$:

$$g(\rho) = \sum_{I_{oo} \cup I_s \cup \{i_{k+1}, \ldots, i_m\}} w_i|\lambda_i| - \sum_{\{i_1, \ldots, i_k\}} w_i|\lambda_i|$$

$$+ \rho \left\{ \sum_{I_o \cup I_s \cup \{i_1, \ldots, i_k\}} w_i|\alpha_i| - \sum_{\{i_{k+1}, \ldots, i_n\}} w_i|\alpha_i| \right\}$$

for $k = 1,2,\ldots,n-1$.

For

$\rho \geqslant \rho_m$:

$$g(\rho) = \sum_{I_{oo} \cup I_s} w_i|\lambda_i| - \sum_{I_d} w_i|\lambda_i| + \sum_{I_o \cup I_s \cup I_d} w_i|\alpha_i|$$

Evidently $g(\rho)$ is linear in each interval $\rho_k \leqslant \rho < \rho_{k+1}$ $(k=0,1,\ldots,n-1)$ and $\rho \geqslant \rho_n$ with increasing coefficient of ρ as ρ increases. I.e.,

$$g(\rho) = \begin{cases} t_k + s_k\rho \quad , & \rho_k \leqslant \rho < \rho_{k+1} \quad , \quad k = 0,\dots, n-1 \\ t_n + s_n\rho \quad , & \rho \geqslant \rho_n \end{cases}$$

where $s_o \leq s_1 \leq \dots \leq s_{n-1} \leq s_n$, $s_n = \sum_i w_i|\alpha_i| > 0$ and

$g(\rho_{k+1}) = t_{k+1} + s_{k+1}\,\rho_{k+1} = t_k + s_k\,\rho_{k+1}$.

Let r be the least integer for which $s_r > 0$.

Then $g(\rho_o) \geq g(\rho_1) \geq \dots \geq g(\rho_r)$ and $g(\rho_r) \leq g(\rho_{r+1}) \leq \dots \leq g(\rho_n)$, i.e. $g(\rho_r)$ is the minimum of $g(\rho)$ for $\rho \geq 0$. For any $t \in F$, $t \geq g(\rho_r)$, either $g(\rho_k) \leq t \leq g(\rho_{k+1})$ for some $k \geq r$, or else $t \geq g(\rho_n)$. Thus, $g(\rho) = t$ for either $\rho = (t + t_k)/s_k$ or else for $\rho = (t - t_n)/s_n$.

Q.E.D.

From this lemma, it is easy to obtain the following theorem.

Theorem 3.1

Suppose $\hat{\Lambda}_w$ has at least two points. If $\lambda \in \hat{\Lambda}_w$ and $\sum_i w_i|\lambda_i| < U$, then λ is a convex combination of $\lambda^1, \lambda^2 \in \hat{\Lambda}_w$ with $\sum_i w_i|\lambda_i{}^1| = U = \sum_i w_i|\lambda_i{}^2|$.

Proof:

Suppose $\lambda' \in \hat{\Lambda}_w$ and $\lambda' \neq \lambda$.

Let $\alpha = \lambda - \lambda' \neq 0$, $g_\alpha(\rho) = \sum_i w_i\, |\lambda_i + \rho\alpha_i|$.

By Lemma 3.1, there exists $\rho_1 > 0$, such that $g_\alpha(\rho_1) = U > g_\alpha(0)$.

Set

$$\lambda' = \lambda + \rho_1\alpha \text{ .}$$

Since $\sum_i P_i\lambda'_i = \sum_i P_i\ (\lambda_i + \rho_1\,\alpha_i) =$

$$= \sum_i P_i\ (\lambda_i + \rho_1\ (\lambda_i - \lambda'_i)) = Q + \rho_1\ (Q - Q) = Q$$

and

$$\sum w_i|\lambda'_i| = g\alpha(\rho_1) = U,$$

therefore

$$\lambda' \in \hat{\Lambda}w$$

Similarly, there exists $\rho_2 > 0$ such that

$$g_{-\alpha}(\rho_2) = \sum_i w_i|\lambda_i + \rho_2 (-\alpha_i)| = U$$

Now $\lambda^2 \stackrel{\Delta}{=} \lambda + \rho_2 (-\alpha) = \lambda - \rho_2\alpha \quad \hat{\Lambda}_w$.

Thus $\lambda = \dfrac{\rho_2}{\rho_1 + \rho_2} \lambda^1 + \dfrac{\rho_1}{\rho_1 + \rho_2}$

Q.E.D.

Consider the following set

(7) $$\Lambda_w^{+-} = \left\{ (\lambda^+,\lambda^-) \in F^{(I)} \times F^{(I)} : \sum_i P_i\lambda_i^+ + \sum_i (-P_i)\lambda_i^- = Q, \lambda_i^+\lambda_i^- \geq 0 \right.$$
$$\left. \sum_i w_i\lambda_i^+ + \sum_i w_i\lambda_i^- = U \right\}$$

where $w_i > 0, \forall i \in I$.

Clearly, $\left\{ \begin{pmatrix} P_i \\ w_i \end{pmatrix}, \begin{pmatrix} -P_i \\ w_i \end{pmatrix} : i \in I \right\}$ has the opposite sign property.

Hence, Λ_w^{+-} is the convex hull of its extreme points. Furthermore, we have the following:

Lemma 3.2

If (λ^+, λ^-) is an extreme point of Λ_w^{+-} , and $\lambda_i^+ \lambda_i^- = 0$ holds for all $i \in 1$, then $\lambda = \lambda^+ - \lambda^-$ is an extreme point of $\hat{\Lambda}_w$.

Proof:

Since $\lambda_i^+ \lambda_i^- = 0$ and $\lambda_i^+ \geq 0$, $\lambda_i^- \geq 0$

$$(8) \qquad \sum_i w_i|\lambda_i| = \sum_i w_i \lambda_i^+ + \sum_i w_i \lambda_i^- = U$$

Also $\sum_i P_i\lambda_i = \sum_i P_i\lambda_i^+ + \sum_i (-P_i)\lambda_i^- = Q$

Thus, $\lambda \in \hat{\Lambda}_w$, if there are λ^1, $\lambda^2 \in \hat{\Lambda}_w$ such that

$$\lambda = \theta \lambda' + (\overline{1-\theta}) \lambda^2, \text{ where } 1 > \theta > 0.$$

If there exists an i_o such that $\text{sgn}\lambda_{i_o}' \neq \text{sgn}\lambda_{i_o}^2$, then

$$|\lambda_{i_o}| = |\theta\lambda_{i_o}^1 + (1-\theta) \lambda_{i_o}^2| < \theta |\lambda_{i_o}^1| + (1-\theta) |\lambda_{i_o}^2|$$

Therefore $\sum_i w_i|\lambda_i| < \theta \sum_i w_i|\lambda_i^1| + (1-\theta) \sum_i w_i|\lambda_i^2| \leq U$

This is a contradiction to (8), so

$$\text{sgn } \lambda_i^1 = \text{sgn } \lambda_i^2$$

and

$$\sum_i w_i|\lambda_i^1| = U \quad , \quad \sum_i w_i|\lambda_i^2| = U$$

For $k = 1,2$, set

$$\lambda_i^{k+} = \begin{cases} \lambda_i^k & \text{if } \lambda_i^k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i^{k-} = \begin{cases} -\lambda_i^k & \text{if } \lambda_i^k < 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence $(\lambda^+,\lambda^-) = \theta(\lambda^{1+}, \lambda^{1-}) + (1-\theta)\ (\lambda^{2+}, \lambda^{2-})$

with $(\lambda^{k+}, \lambda^{k-}) \in \hat{\Lambda}_W^{+-}$, $k=1,2$

Recalling (λ^+, λ^-) is an extreme point of Λ_W^{+-}, there must hold

$$(\lambda^+,\lambda^-) = (\lambda^{1+}, \lambda^{1-}) = (\lambda^{2+}, \lambda^{2-}).$$

Thereby $\lambda = \lambda^1 = \lambda^2$ Q.E.D.

Theorem 3.2

Every $\lambda \in \hat{\Lambda}_W$ with $\sum_i w_i|\lambda_i| = U$ is a convex combination of extreme points of $\hat{\Lambda}_W$.

Proof:

Let $\lambda^+,\lambda^- \in F^{(I)}$ such that

$$\lambda_i^+ = \begin{cases} \lambda_i & \lambda_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i^- = \begin{cases} -\lambda_i & \lambda_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $\lambda \in \hat{\Lambda}_W$ with $\sum_i w_i|\lambda_i| = U$ and $\lambda = \lambda^+ - \lambda^-$, $(\lambda^+, \lambda^-) \in \Lambda_W^{+-}$

If (λ^+,λ^-) is not an extreme point of Λ_W^{+-} , then it is the convex combination of extreme points of Λ_W^{+-} , i.e.

$$(\lambda^+, \lambda^-) = \sum_k \theta_k\ (\lambda^{+k}, \lambda^{-k})$$

where $\theta_k > 0 \quad \sum_k \theta_k = 1$

Because $0 = \lambda_i^+ \lambda_i^- \geq \sum_k \theta_k^2 \lambda_i^{+k} \lambda_k^{-k} \geq 0$ these extreme points $(\lambda^{+k}; \lambda^{-k})$ must have the property that

$$\lambda_i^{+k} \cdot \lambda_i^{-k} = 0$$

By lemma 3.2, they correspond via $\lambda^k = \lambda^{+k} - \lambda^{-k}$ to extreme points of $\hat{\Lambda}_w$ and

$$\lambda = \sum_k \theta_k \lambda^k .$$

Q.E.D.

From theorem 3.1 and 3.2, the following corollary holds.

Corollary: $\hat{\Lambda}_w$ is the convex hull of its extreme points.

Since $\hat{\Lambda}$ is a special case of $\hat{\Lambda}_w$, $\hat{\Lambda}$ is also the convex hull of its extreme points. Furthermore, we have the following theorem that gives characteristic properties of the extreme points of $\hat{\Lambda}$.

Theorem 3.3

If $\hat{\Lambda}$ has at least two points and $\lambda \in \hat{\Lambda}$, then λ is an extreme point of $\hat{\Lambda}$ if and only if

i) $\sum_i |\lambda_i| = U$

ii) $\left\{ P_i : \lambda_i > 0 \right\} \cup \left\{ -P_i : \lambda_i < 0 \right\}$ is affinely independent.

Proof:

Suppose that λ is an extreme point of $\hat{\Lambda}$. By theorem 3.1 $\sum_i |\lambda_i| = U$.

Let $\lambda_i^+ = \begin{cases} \lambda_i & \lambda_i > 0 \\ 0 & \text{otherwise} \end{cases}$

$\lambda_i^- = \begin{cases} -\lambda_i & \lambda_i < 0 \\ 0 & \text{otherwise} \end{cases}$

Thus $(\lambda^+, \lambda^-) \in \hat{\Lambda}^{+-}$

where

$$\hat{\Lambda}^{+-} = \left\{ (\lambda^+, \lambda^-) \in F^{(I)} \times F^{(I)} : \begin{array}{l} \sum_i P_i \lambda_i^+ + \sum_i (-P_i) \lambda_i^- = Q, \lambda_i^+, \lambda_i^- \geq 0 \\ \sum_i \lambda_i^+ + \sum_i \lambda_i^- = U \end{array} \right\}$$

It is easy to verify that (λ^+, λ^-) must be an extreme point of $\hat{\Lambda}^{+-}$. By theorem 1.1, the LIEP theorem,

$$\left\{ \begin{pmatrix} P_i \\ 1 \end{pmatrix} : \lambda_i^+ > 0 \right\} \cup \left\{ \begin{pmatrix} -P_i \\ 1 \end{pmatrix} : \lambda_i^- > 0 \right\}$$

is linearly independent. In other words,

$$\left\{ P_i : \lambda_i > 0 \right\} \cup \left\{ -P_i : \lambda_i < 0 \right\}$$

is affinely independent.

Suppose that (i), (ii) hold for some $\lambda \in \hat{\Lambda}$. By the same transformation, we have

$$\lambda = \lambda^+ - \lambda^-$$

with $(\lambda^+, \lambda^-) \in \hat{\Lambda}^{+-}$ and $\lambda_i^+ \lambda_i^- = 0, \ \forall_i$

By the LIEP theorem also, (λ^+, λ^-) is an extreme point of $\hat{\Lambda}^{+-}$. Finally, the lemma 3.2 ensures that λ is an extreme point of $\hat{\Lambda}$.

Q.E.D.

4. The set Λ^c

If I is an infinite index set, it is interesting to see following results.

Theorem 4.1

If I has an infinite number of elements, then Λ^c has no extreme point at all.

Proof: We only need to show that for any $\lambda \in \Lambda^c$, there exist $\lambda^1, \lambda^2 \in \Lambda^c$, $\lambda_1 \neq \lambda_2$ such that

$$\lambda = \tfrac{1}{2}\lambda^1 + \tfrac{1}{2}\lambda^2$$

Since $\{i \in I : \lambda_i \neq 0\}$ only has a finite number of elements, we can select $i_1,\ldots,i_{m+1} \in I$, such that

$$\{i_1,\ldots,i_{m+1}\} \cap \{i \in I : \lambda_i \neq 0\} = \phi$$

Because $P_i ,\ldots,P_{i_{m+1}} \in F^m$, there exist $\alpha_1,\ldots,\alpha_{m+1}$ not all zero, such that

$$\sum_{k=1}^{m+1} P_{i_k}\alpha_k = 0$$

Let

$$\theta = \min \{U/|\alpha_k| : \alpha_k \neq 0\}$$

$$\lambda_i^1 = \begin{cases} \lambda_i & \text{if } \lambda_i \neq 0 \\ \theta\alpha_k & \text{if } i = i_k \text{ , } k = 1,\ldots,m+1 \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda_i^2 = \begin{cases} \lambda_i & \text{if } \lambda_i \neq 0 \\ -\theta\alpha_k & \text{if } i = i_k \text{ , } k = 1,\ldots,m+1 \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$\lambda^1, \lambda^2 \in \Lambda^C$$

Clearly $\lambda^1 \neq \lambda^2$ and

$$\lambda = \tfrac{1}{2}\lambda^1 + \tfrac{1}{2}\lambda^2$$

Q.E.D.

The basis for this contrast with the "(ℓ_1)" results is that the latter involves only a finite number of additional constraints, whereas the Chebychev bounds involve the whole infinite cardinality of bounds corresponding to the totality of coordinates. However, every point in Λ^C has only a finite number of non-zero coordinates. And, for the projection of Λ^C onto the finite-dimensional subspace corresponding to these, we do have extreme point theorems of similar nature to the (ℓ_1) ball results as we now show.

We will now develop a generalization of the semi-infinite LIEP Theorem which perhaps shows why Λ^C has no extreme points at all. Actually Theorem 4.1 can be regarded as a corollary of the following Theorem 4.2.

Define the set

(9) $$\Lambda' \triangleq \{\lambda \in F^{(I)} : \sum_i P_i \lambda_i = Q \ , \ \ell_i \leq \lambda_i \leq u_i \ , \ \forall i\}$$

where $\ell_i \leq u_i$.

Note that since $\lambda \in F^{(I)}$, $\Lambda' = \phi$ unless $\ell_i \leq 0$, $\forall i \in I$.

<u>Theorem 4.2</u>

$\lambda \in \Lambda' \neq \phi$ is an extreme point of Λ' if and only if $\{P_i : \ell_i < \lambda_i < u_i\}$ is linearly independent.

<u>Proof</u>: <u>"if"</u>: Suppose $\{P_i : \ell_i < \lambda_i < u_i\}$ is linearly independent and

(10) $$\lambda = \theta\lambda^1 + (1 - \theta)\lambda^2$$

where $0 < \theta < 1$, $\lambda^1, \lambda^2 \in \Lambda'$.

Let

$$I_\ell \triangleq \{i \in I : \lambda_i = \ell_i\}$$

$$I_u \triangleq \{i \in I : \lambda_i = u_i\}$$

$$I_{oo} \triangleq \{i \in I : \ell_i < \lambda_i < u_i\}$$

Clearly, from (10), we have

$$i \in I_\ell \Rightarrow \lambda_i^1 = \lambda_i^2 = \ell_i$$

$$i \in I_u \Rightarrow \lambda_i^1 = \lambda_i^2 = u_i$$

Thus,

$$\sum_{i \in I_{oo}} P_i \lambda_i = Q - \sum_{i \in I} P_i \ell_i - \sum_{i \in I_u} P_i u_i = \sum_{i \in I_{oo}} P_i \lambda_i^1 = \sum_{i \in I_{oo}} P_i \lambda_i^2$$

But since $\{P_i : i \in I_{oo}\}$ is linearly independent, the representation is unique.

Hence

$$\lambda_i = \lambda_i^1 = \lambda_i^2$$

i.e., $\lambda = \lambda^1 = \lambda^2$ and λ is an extreme point.

<u>"only if"</u>: Suppose $\{P_i : i \in I_{oo}\}$ is linearly dependent, i.e.,

$$\sum_{i \in I_{oo}} P_i \alpha_i = 0$$

with not all $\alpha_1, \ldots, \alpha_n$ zero.

Take $\varepsilon > 0$ small enough so that

$$\ell_i < \lambda_i + \varepsilon\alpha_i < u_i \qquad \forall i \in I_{oo}$$

Let

$$\lambda_i^1 \triangleq \begin{cases} \lambda_i + \varepsilon\alpha_i, & i \in I_{oo} \\ \lambda_i, & \text{otherwise} \end{cases}$$

$$\lambda^2 \triangleq \begin{cases} \lambda_i - \varepsilon\alpha_i \ , & i \in I_{oo} \\ \lambda_i & , \text{otherwise} \end{cases}$$

Evidently, $\lambda^1, \lambda^2 \in \Lambda'$ $\lambda^1 \neq \lambda^2$ and

$$\lambda = \tfrac{1}{2}\lambda^1 + \tfrac{1}{2}\lambda^2$$

Q.E.D.

Since Λ^c is a special case of Λ' with $u_i = U$, $\ell_i = -U$, and Λ^c is in a generalized finite sequence space,

$$I_{oo} = \{i \in I : -u_i < \lambda_i < U\} \supseteq \{i \in I : \lambda_i = 0\}$$

has an infinite number of elements. Thus $\{P_i : i \in I_{oo}\}$ must be linearly dependent for any $\lambda \in \Lambda^c$. By Theorem 4.2, there is no extreme point in Λ^c.

However if all $\ell_i = 0$, Λ' is the convex hull of its extreme points and we have the following Theorem 4.3.

First for simpler discourse, we shall call

$$I_{oo} = \{i \in I : 0 < \lambda_i < u_i\}$$

the "active index set" from now on.

<u>Theorem 4.3</u>: Λ' is the convex hull of its extreme points, if all $\ell_i = 0$.

<u>Proof</u>: Take any $\lambda \in \Lambda'$. If λ is not an extreme point of Λ', by Theorem 4.2, $\{P_i : i \in I_{oo}\}$ is linearly dependent. Thus, there exist α_i, $i \in I_{oo}$ not all zero such that

$$\sum_{i \in I_{oo}} P_i\alpha_i = 0$$

Let

$$I_{oo}^{+} = \{i \in I_{oo} : \alpha_i > 0\}$$

$$I_{oo}^{-} = \{i \in I_{oo} : \alpha_i < 0\}$$

$$\rho_1 = \min \left\{ \frac{u_i - \lambda_i}{\alpha_i} \; ; \; i \in I_{00}^+ \; ; \; \frac{\lambda_i}{|\alpha_i|} \; , \; i \in I_{00}^- \right\}$$

$$\rho_2 = \min \left\{ \frac{\lambda_i}{\alpha_i} \; , \; i \in I_{00}^+ \; ; \; \frac{u_i - \lambda_i}{|\alpha_i|} \; , \; i \in I_{00}^- \right\}$$

Because $I_{00}^+ \cup I_{00}^- \neq \phi$, ρ_1 and ρ_2 are well defined. Now let

$$\lambda_i^1 \triangleq \begin{cases} \lambda_i & , \; i \notin I_{00}^+ \cup I_{00}^- \\ \lambda_i + \rho_1 \alpha_i & , \; i \in I_{00}^+ \cup I_{00}^- \end{cases}$$

$$\lambda_i^2 \triangleq \begin{cases} \lambda_i - \rho_2 \alpha_i & , \; i \in I_{00}^+ \cup I_{00}^- \\ \lambda_i & , \; \text{otherwise} \end{cases}$$

It is easy to see that

$$\lambda^1 , \; \lambda^2 \in \Lambda' \; , \; \lambda^1 \neq \lambda^2 \; ,$$

$$\lambda = \frac{\rho_2}{\rho_1 + \rho_2} \lambda^1 + \frac{\rho_2}{\rho_1 + \rho_2} \lambda^2$$

and the active index set of λ^k $(k = 1,2)$ has at least one less element than the active index set of λ. If λ^1 and λ^2 are both extreme points of Λ', we are done. Otherwise using the same method, we can present λ^k as a convex combination of two other points of Λ' which have at least one less element of the active index set than the λ^k's. Therefore in at most 2^n steps (where n is the number of elements of I_{00}) we can get λ as a convex combination of extreme points of Λ'.

Q.E.D.

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The extreme point characterization of the (ℓ')-ball of a generalized finite sequence space by Kortanek and Strojwas was accomplished only for real scalars and by continuity considerations. We show that no topology or continuity is needed as in Kortanek-Strojwas and that the characterization extends to weighted (ℓ')-balls with any ordered field. We show a Chebyshev ball theorem is false since they have no extreme points. Via generalizing the LIEP theorem, useful projections of the ball are proved convex hulls of their extreme points.

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